CENTRAL LIMIT THEOREM FOR ARTIN L-FUNCTIONS

PETER J. CHO AND HENRY H. KIM*

Abstract. We show that the sum of the traces of Frobenius elements of Artin L-functions in a family of G-fields satisfies the Gaussian distribution under certain counting conjectures. We prove the counting conjectures for S_4 and S_5-fields. We also show central limit theorem for modular form L-functions with the trivial central character with respect to congruence subgroups as the level goes to infinity.

1. Introduction

Let G be a finite group which is a transitive subgroup of a certain symmetry group S_{d+1}. A number field K of degree $d+1$ is called a G-field if its Galois closure $\widehat{K}$ over $\mathbb{Q}$ is a G-Galois extension. For a G-field K, we attach the Artin L-function

$$L(s,\rho,K)=\frac{\zeta_K(s)}{\zeta(s)}=\sum_{n=1}^{\infty}a_\rho(n)n^{-s},$$

where ρ is d-dimensional representation of G. Note that $-1\le a_\rho(p)\le d$. If $G=S_{d+1}$, ρ is the d-dimensional standard representation of S_{d+1}. Let $L(X)^{r_2}$ be the set of G-fields K with $|d_K|<X$ and signature (r_1,r_2). In this paper, we restrict to the case $G=S_{d+1}$, and consider the sum $\sum_{p\le x}a_\rho(p)$ in the family of S_{d+1}-fields, $L(X)^{r_2}$ and show that under the counting conjectures (2.1) and (2.2), it has the Gaussian distribution, namely, for a continuous real function h on $\mathbb{R}$, if $\frac{\log X}{\log x}\to\infty$ as $x\to\infty$,

$$\frac{1}{\#L(X)^{r_2}}\sum_{L(s,\rho,K)\in L(X)^{r_2}}h\left(\frac{\sum_{p\le x}a_\rho(p)}{\sqrt{\pi(x)}}\right)\longrightarrow\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(t)e^{-\frac{t^2}{2}}\,dt. \tag{1.1}$$

When $L(X)^{r_2}$ is replaced by $\mathcal{F}_k$, the family of all normalized holomorphic Hecke eigen cusp forms of weight k with respect to $SL_2(\mathbb{Z})$, Nagoshi [13] showed that if $\frac{\log k}{\log x}\to\infty$ as $x\to\infty$,

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$$\frac{1}{\#\mathcal{F}_k}\sum_{f\in\mathcal{F}_k} h\left(\frac{\sum_{p\le x} a_f(p)}{\sqrt{\pi(x)}}\right) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}}\,dt, \tag{1.2}$$

and called it central limit theorem.

Here we note that we do not need the Artin conjecture nor the strong Artin conjecture in the proof of (1.1). The estimates (2.1) and (2.2) are proved by Taniguchi and Thorne [17] for S_3 fields. For $G = S_4, S_5$, the estimate (2.1) was proved in [3] and [16], resp. We prove (2.2) in Sections 6 and 7. Hence (1.1) is unconditional for S_3, S_4 and S_5-fields. These estimates will be used in computing the n-level densities of Artin L-functions [7], [8].

We also study the distribution of the prime sums $\sum_{p\le x} a_\rho(p)^r$ for a positive integer r. The effective Chebotarev density theorem implies the analogue of Sato-Tate distribution. Namely, $\frac{1}{\pi(x)}\sum_{p\le x} a_\rho(p)^r \to n_r$ as $x\to\infty$, where n_r is the multiplicity of the trivial representation in ρ^r.

We also study distribution of $\sum_{p\le x} a_f(p)$ for $f \in S_k(N)$, the set of normalized Hecke eigen cusp forms of weight k with respect to $\Gamma_0(N)$ with the trivial central character. We prove the Gaussian distribution as $N\to\infty$.

2. Counting S_{d+1}-fields with local conditions

Let $\mathcal{S} = (LC_p)$ be a finite set of local conditions. $LC_p = \mathcal{S}_{p,C}$ means that p is unramified in K and the conjugacy class of Frob_p is C. Let $|\mathcal{S}_{p,C}| = \frac{|C|}{|G|(1+f(p))}$ for some positive function $f(p)$ which satisfies $f(p) = O(\frac{1}{p})$. There are also several splitting types of ramified primes, which are denoted by $r_1, r_2, \cdots, r_w$. If $LC_p = \mathcal{S}_{p,r_i}$, then it means that p is ramified and its splitting type is r_i. Assume that we can choose explicit positive functions $c_1(p), c_2(p), \cdots, c_w(p)$ with $\sum_{i=1}^w c_i(p) = f(p)$. Define $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$ and $|\mathcal{S}| = \prod_p |LC_p|$.

Let $L(X;\mathcal{S})^{r_2}$ be the set of S_{d+1}-fields K with $|d_K| < X$ and the local condition $\mathcal{S}$.

Then assume that

$$|L(X)^{r_2}| = A(r_2)X + O(X^\delta), \tag{2.1}$$

$$|L(X;\mathcal{S})^{r_2}| = |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p\in S} p\right)^\gamma X^\delta\right), \tag{2.2}$$

for some positive constant δ and γ, and the implied constant is uniformly bounded for p and local conditions at p.

This assumption is satisfied when $G = S_3, S_4$ and S_5. When $G = S_3$, Taniguchi and Thorne [17] obtained more precise results: Let $L(X)^{\pm}$ be the set of cubic fields K with $\pm d_K < X$. Then

$$|L(X)^{\pm}| = \frac{C^{\pm}}{12\zeta(3)}X + \frac{4K^{\pm}}{5\Gamma(2/3)^3}X^{5/6} + O(X^{7/9+\epsilon}),$$

where $C^- = 3$, $C^+ = 1$, $K^- = \sqrt{3}$, and $K^+ = 1$. Here, we count only one cubic field from three conjugate fields. Let TS_p, PS_p, and IN_p be the local conditions of p which means that p is totally split, partially split and inert respectively. Let $S = \{LC_{p_i} | i = 1, 2, \cdots, u\}$ be a set of local conditions at p_i. Then

$$|LC_p| = \begin{cases} \frac{1/6}{1+1/p+1/p^2} & \text{if } LC_p = TS_p, \\ \frac{3/6}{1+1/p+1/p^2} & \text{if } LC_p = PS_p, \\ \frac{2/6}{1+1/p+1/p^2} & \text{if } LC_p = IN_p, \\ \frac{1/p}{1+1/p+1/p^2} & \text{if } p \text{ is partially ramified,} \\ \frac{1/p^2}{1+1/p+1/p^2} & \text{if } p \text{ is totally ramified.} \end{cases}$$

and $|L(X; \mathcal{S})^{\pm}| = |\mathcal{S}|A^{\pm}X + O(E_{\mathcal{S}}(X))$, where $A^{\pm} = C_1 \frac{C^{\pm}}{12\zeta(3)}$, and

$$E_{\mathcal{S}}(X) = \begin{cases} X^{5/6} & \text{if } (\prod_{p \in \mathcal{S}} p^{\frac{8}{9}e_p}) < X^{\frac{1}{16}}, \\ (\prod_{p \in \mathcal{S}} p^{\frac{8}{9}e_p}) X^{\frac{7}{9}+\epsilon} & \text{if } (\prod_{p \in \mathcal{S}} p^{\frac{8}{9}e_p}) \geq X^{\frac{1}{16}}, \end{cases}$$

where $e_p = 1$ if p is unramified and otherwise, $e_p = 2$.

For $G = S_4, S_5$, the estimate (2.1) was proved in [3] and [16]. We prove (2.2) in Sections 6 and 7.

We identify $L(X)^{r_2}$ with the set of Artin L-functions $L(s, \rho, K)$ where K is a G-field with $|d_K| < X$ with signature (r_1, r_2). Here we count only one G-field for each $d+1$ conjugate fields. Throughout the article, we implicitly assume that the size of $L(X)^{r_2}$ is same with the number of number fields which satisfy the requirements of $L(X)^{r_2}$. This claim deals with arithmetic equivalence of number fields. Two number fields K_1 and K_2 are arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. We say that a number field K is arithmetically solitary if $\zeta_K(s) = \zeta_F(s)$ implies that K and F are conjugate. It is known that S_d-fields and A_d fields are arithmetically solitary. See Chapter II in [12].

For simplicity, we denote $L(s, \rho, K) \in L(X)^{r_2}$ by $\rho \in L(X)$.

3. Central limit theorem of Artin L-functions

Consider, for a continuous real function h on $\mathbb{R}$,

$$\frac{1}{\#L(X)}\sum_{\rho\in L(X)} h\left(\frac{\sum_{p\le x} a_\rho(p)}{\sqrt{\pi(x)}}\right). \tag{3.1}$$

We assume that x grows more slowly than X; namely, $\frac{\log X}{\log x}\longrightarrow\infty$ as $x\to\infty$. So for an arbitrary positive real number a, we have $X > x^a$.

By Theorem 25.8 and Theorem 30.2 (the method of moments) in [1], it is enough to consider $h(x)=x^r$. Consider

$$\sum_{\rho\in L(X)}\left(\frac{\sum_{p\le x} a_\rho(p)}{\sqrt{\pi(x)}}\right)^r. \tag{3.2}$$

By multinomial formula,

$$\left(\sum_{p\le x} a_\rho(p)\right)^r=\sum_{u=1}^r {\sum}^{(1)}_{(r_1,...,r_u)}\frac{r!}{r_1!\cdots r_u!}\frac{1}{u!}{\sum}^{(2)}_{(p_1,...,p_u)} a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u},$$

where $\sum^{(1)}_{(r_1,...,r_u)}$ means the sum over the u-tuples $(r_1,...,r_u)$ of positive integers such that $r_1+\cdots+r_u=r$, and $\sum^{(2)}_{(p_1,...,p_u)}$ means the sum over the u-tuples $(p_1,...,p_u)$ of distinct primes such that $p_i\le x$ for each i. Then

$$(3.2)=\pi(x)^{-\frac{r}{2}}\sum_{u=1}^r\frac{1}{u!}{\sum}^{(1)}_{(r_1,...,r_u)}\frac{r!}{r_1!\cdots r_u!}{\sum}^{(2)}_{(p_1,...,p_u)}\left(\sum_{\rho\in L(X)} a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u}\right).$$

Now we claim that except when r is even, $u=\frac{r}{2}$, and $r_1=\cdots=r_u=2$, it gives rise to the error term.

Now suppose $r_i\ge 2$ for all i, and $r_j>2$ for some j. Then since $r_1+\cdots+r_u=r$, $u\le\frac{r-1}{2}$. Hence by the trivial estimate, such term is majorized by

$$\pi(x)^{-\frac{r}{2}}\sum_{u=1}^r\frac{1}{u!}{\sum}^{(1)}_{(r_1,...,r_u)}\frac{r!}{r_1!\cdots r_u!}d^{r_1+\cdots+r_u}|L(X)|\pi(x)^u\ll_r\pi(x)^{-\frac{1}{2}}|L(X)|\sum_{u=1}^r\frac{1}{u!}d^r\ll_{r,d}X\pi(x)^{-\frac{1}{2}}.$$

This gives rise to the error term.

Suppose $r_i\le 2$ for all i. Suppose $r_i=1$ for some i. We may assume that $r_1=1$.

Let N be the number of conjugacy classes of G, and partition the sum $\sum_{\rho\in L(X)}$ into $(N+w)^u$ sums, namely, given $(\mathcal{S}_1,...,\mathcal{S}_u)$, where $\mathcal{S}_i$ is either $\mathcal{S}_{p_i,C}$ or $\mathcal{S}_{p_i,r_j}$, we consider the set of $\rho\in L(X)$ with the local conditions $\mathcal{S}_i$ for each i. Note that in each such partition, $a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u}$ remains a constant.

Suppose p_1 is unramified, and fix the splitting types of $p_2, \cdots, p_u$, and let Frob_{p_1} runs through the conjugacy classes of G. Then by (2.2), the sum of such N partitions is

$$\sum_C \left(\frac{|C| a_\rho(p_1)}{|G|(1+f(p_1))} A(\mathcal{S}_2, ..., \mathcal{S}_u) X + O((p_1 \cdots p_u)^\gamma X^\delta) \right),$$

for a constant $A(\mathcal{S}_2, ..., \mathcal{S}_u)$. Let χ_ρ be the character of ρ. Then $a_\rho(p) = \chi_\rho(g)$, where $g = \mathrm{Frob}_p$. By orthogonality of characters, $\sum_C |C| a_\rho(p_1) = \sum_{g \in G} \chi_\rho(g) = 0$. Hence the above sum is $O((p_1 \cdots p_u)^\gamma X^\delta)$, and it is majorized by $\pi(x)^{-\frac{r}{2}+u} x^{\gamma u} X^\delta$.

Hence we can assume that $r_i \leq 2$ for each i, and p_j is ramified when $r_j = 1$. Suppose $r_1 + \cdots + r_v + r_{v+1} + \cdots + r_u = r$, $r_1 = \cdots = r_v = 1$ and $r_{v+1} = \cdots = r_u = 2$. Then $u - v \leq \frac{r-1}{2}$, and $p_1, ..., p_v$ are ramified. The partition of fixed splitting types of $p_{v+1}, ..., p_u$ is majorized by

$$\prod_{i=1}^{v} \frac{f(p_i)}{1+f(p_i)} B(\mathcal{S}_{v+1}, ..., \mathcal{S}_u) X + O((p_1 \cdots p_u)^\gamma X^\delta),$$

for some constant $B(\mathcal{S}_{v+1}, ..., \mathcal{S}_u)$. Since $\frac{f(p)}{1+f(p)} \ll \frac{1}{p}$, it contributes to

$$\pi(x)^{u-v-\frac{r}{2}} (\log\log x)^v X + \pi(x)^{-\frac{r}{2}+u} x^{\gamma u} X^\delta \ll X (\log\log x)^v \pi(x)^{-\frac{1}{2}} + \pi(x)^{-\frac{r}{2}+u} x^{\gamma u} X^\delta.$$

Now let r be even, $u = \frac{r}{2}$, and $r_1 = \cdots = r_u = 2$. If one of $p_1, p_2, \cdots, p_u$ is ramified, their contribution is majorized by $X \pi(x)^{-1} \log\log x$. Now we assume that all primes are unramified. Then the corresponding term is

$$\pi(x)^{-\frac{r}{2}} \frac{1}{u!} \frac{r!}{2^u} \sum_{(p_1, ..., p_u)}^{(2)} \left(\sum_{L(s,\rho) \in L(X)} a_\rho(p_1)^2 \cdots a_\rho(p_u)^2 \right). \tag{3.3}$$

Let N be the number of conjugacy classes of G, and partition the sum $\sum_{\rho \in L(X)}$ into N^u sums where $(C_1, ..., C_u)$ is the set of $\rho \in L(X)$ such that $\mathrm{Frob}_{p_i} \in C_i$ for each i. Then,

$$\sum_{\rho \in L(X)} a_\rho(p_1)^2 \cdots a_\rho(p_u)^2 = \sum_{(C_1, ..., C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \left(\sum_{\substack{\rho \in L(X) \\ \mathrm{Frob}_{p_i} \in C_i}} 1 \right)$$

$$= \sum_{(C_1, ..., C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \left(\prod_{i=1}^{u} \frac{|C_i|}{|G|(1+f(p_i))} |L(X)| + O((p_1 \cdots p_u)^\gamma X^\delta) \right).$$

Now

$$\sum_{(C_1, ..., C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \prod_{i=1}^{u} \frac{|C_i|}{|G|(1+f(p_i))} = \prod_{i=1}^{u} \left(\sum_{C_i} \frac{\chi_\rho(p_i)^2 |C_i|}{|G|(1+f(p_i))} \right).$$

Here $\chi_\rho(p)^2 = \chi_{\rho^2}(p) = \chi_{Sym^2\rho}(p) + \chi_{\wedge^2\rho}(p)$. We observed in [7] that since ρ is an irreducible real self-dual representation, $Sym^2\rho$ contains the trivial representation and $\wedge^2\rho$ does not contain the trivial representation ([11], page 274). Hence $\chi_\rho(p)^2 = 1 + \sum_{j=1}^{l} \eta_j(p)$, where η_j's are non-trivial irreducible characters of G. By the orthogonality of characters, for each j, $\sum_C |C|\eta_j(p) = \sum_{g\in G} \eta_i(g) = 0$. Hence $\sum_C \chi_\rho(p)^2|C| = |G|$. Therefore,

$$\begin{aligned} &\sum\nolimits^{(2)}_{(p_1,\ldots,p_u)} \left(\sum_{\rho\in L(X)} a_\rho(p_1)^2 \cdots a_\rho(p_u)^2 \right) \\ = \; &\pi(x)^u |L(X)| + O(\pi(x)^{u-1}|L(X)|\log\log x) + O(\pi(x)^u x^{\gamma u} X^\delta). \end{aligned}$$

Note

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}}\, dt = \begin{cases} \frac{r!}{(r/2)!2^{r/2}}, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd} \end{cases}.$$

Hence we have proved

Theorem 3.1. *Suppose* $\frac{\log X}{\log x} \longrightarrow \infty$ *as* $x \to \infty$. *Then*

$$\frac{1}{|L(X)|} \sum_{\rho\in L(X)} \left(\frac{\sum_{p\le x} a_\rho(p)}{\sqrt{\pi(x)}} \right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}}\, dt + O\left(\frac{(\log\log x)^r}{\pi(x)^{\frac12}} \right).$$

This proves (1.1).

4. Central Limit Theorem for Hecke eigenforms; Level aspect

In this section, in analogy to (1.2), we consider central limit theorem for modular form L-functions with the trivial central character with respect to congruence subgroups as the level goes to infinity. We follow [13] closely. For $k \ge 2$, let $S_k(N)$ be the set of normalized Hecke eigen cusp forms of weight k with respect to $\Gamma_0(N)$ with the trivial central character. Let $f(z) = \sum_{n=1}^\infty a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$; $a_f(mn) = a_f(m)a_f(n)$, if $(m,n) = 1$; $a_f(1) = 1$; $a_f(p^j) = a_f(p)a_f(p^{j-1}) - a_f(p^{j-2})$.

We show

Theorem 4.1. *For a continuous real function h on $\mathbb{R}$, (assume that* $\frac{\log N}{\log x} \longrightarrow \infty$ *as* $x \to \infty$.*)*

$$\frac{1}{\#S_k(N)} \sum_{f\in S_k(N)} h\left(\frac{\sum_{p\le x} a_f(p)}{\sqrt{\pi(x)}} \right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-\frac{t^2}{2}}\, dt \quad \textit{as } x \to \infty.$$

We have, from [14],

Lemma 4.2. *Suppose $k \geq 2$. Let $S_k(N,\chi)$ be the set of normalized Hecke eigen cusp forms of weight k with respect to $\Gamma_0(N)$ with a character χ (mod N). Then*

$$\sum_{f \in S_k(N,\chi)} a_f(n) = \frac{k-1}{12}\chi(\sqrt{n})n^{-\frac{1}{2}}\psi(N) + O(n^c N^{\frac{1}{2}} d(N)),$$

for some constant c, independent of n, N.

Here $\psi(N) = N\prod_{l|N}(1+\frac{1}{l})$, and $d(N)$ is the number of positive divisors of N. Note that $\psi(N) = |SL_2(\mathbb{Z}) : \Gamma_0(N)|$. Here $\chi(x) = 0$ if x is not a positive integer prime to N. In particular, if n is not a square, $\sum_{f\in S_k(N,\chi)} a_f(n) = O(n^c N^{\frac{1}{2}} d(N))$. Taking $n = 1$ and $\chi = 1$, we have

$$\#S_k(N) = \frac{k-1}{12}\psi(N) + O(N^{\frac{1}{2}} d(N)).$$

We need to compute, for a positive integer r,

$$\sum_{f \in S_k(N)} \left(\frac{\sum_{p \leq x} a_f(p)}{\sqrt{\pi(x)}} \right)^r . \tag{4.1}$$

By multinomial formula,

$$(4.1) = \pi(x)^{-\frac{r}{2}} \sum_{u=1}^{r} \frac{1}{u!} \sum\nolimits^{(1)}_{(r_1,\ldots,r_u)} \frac{r!}{r_1!\cdots r_u!} \sum\nolimits^{(2)}_{(p_1,\ldots,p_u)} \left(\sum_{f\in S_k(N)} a_f(p_1)^{r_1} \cdots a_f(p_u)^{r_u} \right).$$

Now we claim that except when r is even, $u = \frac{r}{2}$, and $r_1 = \cdots = r_u = 2$, it gives rise to the error term.

By [13], Lemma 2, we can show that $a_f(p)^n = \sum_{j=0}^n h_n(j) a_f(p^j)$, where $h_n(j) = 0$ if n is odd and j is even, or if n is even and j is odd. For u-tuples $(r_1, ..., r_u)$ and $(p_1, ..., p_u)$, we define

$$\begin{aligned} A(r_1,...,r_u) &= \sum\nolimits^{(2)}_{(p_1,\ldots,p_u)} B(r_1,...,r_u;p_1,...,p_u), \\ B(r_1,...,r_u;p_1,...,p_u) &= \sum_{f\in S_k(N)} a_f(p_1)^{r_1}\cdots a_f(p_u)^{r_u}. \end{aligned}$$

Then

$$B(r_1,...,r_u;p_1,...,p_u) = \sum_{0\leq j_r \leq r_1,\ldots,0\leq j_u\leq r_u} h_{r_1}(j_1)\cdots h_{r_u}(j_u) \sum_{f\in S_k(N)} a_f(p_1^{j_1}\cdots p_u^{j_u}).$$

As in [13], if r_l is odd for some l, $A(r_1,...,r_u) \ll N^{\frac{1}{2}} d(N)\pi(x)^u x^{cur}$.

Now let $r_1 = \cdots = r_u = 2$. Then r is even, and $u = \frac{r}{2}$.

$$A(r_1, ..., r_u) = \pi(x)^{\frac{r}{2}} \# S_k(N) + O(\pi(x)^{\frac{r}{2}-1} (\log\log x)^{\frac{r}{2}} \# S_k(N)).$$

Now suppose that all r_i's are even, and $r_i > 2$ for some i. Then $u \le \frac{r}{2} - 1$. Then

$$A(r_1, ..., r_u) \ll \pi(x)^{-1} \# S_k(N).$$

Hence, as in Theorem 3.1, we have

Proposition 4.3. *Assume that* $\frac{\log N}{\log x} \longrightarrow \infty$ *as* $x \to \infty$. *Then*

$$\frac{1}{\# S_k(N)} \sum_{f \in S_k(N)} \left(\frac{\sum_{p \le x} a_f(p)}{\sqrt{\pi(x)}} \right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}} \, dt + O\left(\frac{(\log\log x)^{\frac{r}{2}}}{\pi(x)} \right).$$

This proves Theorem 4.1

5. Analogues of Sato-Tate distribution

For a Hecke eigenform $f \in \mathcal{F}_k$, Sato-Tate conjecture says that for a continuous real function h on $[-2, 2]$,

$$\frac{1}{\pi(x)} \sum_{p \le x} h(a_f(p)) \longrightarrow \frac{1}{2\pi} \int_{-2}^{2} h(t) \sqrt{4 - t^2} \, dt, \quad \text{as } x \to \infty.$$

Let $a_f(p) = 2\cos\theta_f(p)$ for $\theta_f(p) \in [0, \pi]$. Then $\{\theta_f(p)\}$ is uniformly distributed with respect to the measure $\frac{2}{\pi} \sin^2\theta \, d\theta$ on $[0, \pi]$. This is proved in [2].

For a vertical Sato-Tate distribution, one can consider, for a fixed prime p,

$$\sum_{f \in \mathcal{F}_k} a_f(p)^n. \tag{5.1}$$

Conrey-Duke-Farmer [9] proved, for a holomorphic form of weight k,

$$\sum_{f \in \mathcal{F}_k} a_f(p)^n = \frac{k}{6\pi} \left(1 + \frac{1}{p} \right) \int_0^{\pi} 2^n \cos^n\theta \frac{\sin^2\theta}{(1 - \frac{1}{p})^2 + \frac{4}{p}\sin^2\theta} \, d\theta + O(p^{\frac{n}{2}+\epsilon}).$$

This implies that $\{\theta_f(p), f \in \mathcal{F}_k\}$ is uniformly distributed with respect to the measure

$$\frac{2}{\pi} \left(1 + \frac{1}{p} \right) \frac{\sin^2\theta}{(1 - \frac{1}{p})^2 + \frac{4}{p}\sin^2\theta} \, d\theta.$$

For Artin L-function analogue of Sato-Tate distribution, we consider, for $r \ge 1$,

$$\frac{1}{\pi(x)} \sum_{p \le x} a_\rho(p)^r. \tag{5.2}$$

In our case, note that $-1 \le a_\rho(p) \le d$. By effective Chebotarev density theorem (cf. [15], page 132), for $\log x \gg |G|(\log \left|d_{\widehat{K}}\right|)^2$,

$$\sum_{\substack{p\le x\\ \mathrm{Frob}_p\in C}} 1 = \frac{|C|}{|G|}\pi(x) + O\left(\pi(x^\beta)\right) + O\left(xe^{-c|G|^{-\frac12}(\log x)^{\frac12}}\right),$$

where β is an exceptional zero of $\zeta_{\widehat{K}}(s)$ such that $1-\beta \le \frac14 \log d_{\widehat{K}}$, if it exists. Hence

$$\sum_{p\le x} a_\rho(p)^r = \sum_C a_\rho(p)^r \left(\sum_{\substack{p\le x\\ \mathrm{Frob}_p\in C}} 1\right) = \sum_C a_\rho(p)^r \frac{|C|}{|G|}\pi(x) + O(\pi(x^\beta) + xe^{-c|G|^{-\frac12}(\log x)^{\frac12}}).$$

Now $\sum_C |C| a_\rho(p)^r = \sum_{g\in G} \chi_\rho(g)^r$ and $\chi_\rho(g)^r = \chi_{\rho^r}(g)$. Note that

$$\frac{1}{|G|}\sum_{g\in G} \chi_{\rho^r}(g) = n_r,$$

which is the multiplicity of the trivial representation in ρ^r. Hence

$$\sum_{p\le x} a_\rho(p)^r = n_r\pi(x) + O(\pi(x^\beta) + xe^{-c|G|^{-\frac12}(\log x)^{\frac12}}).$$

Therefore,

$$\frac{1}{\pi(x)}\sum_{p\le x} a_\rho(p)^r \longrightarrow n_r, \quad \text{as } x\to\infty.$$

For vertical Sato-Tate distribution, for a fixed prime p, consider

$$\frac{1}{|L(X)|}\sum_{\rho\in L(X)} a_\rho(p)^r.$$

Then by (2.2),

$$\sum_{\rho\in L(X)} a_\rho(p)^r = \sum_C a_\rho(p)^r \left(\sum_{\substack{\rho\in L(X)\\ \mathrm{Frob}_p\in C}} 1\right) + a_\rho(p)^r \left(\sum_{\substack{\rho\in L(X)\\ p \text{ is ramified}}} 1\right)$$
$$= \frac{|L(X)|}{|G|(1+f(p))}\sum_C |C| a_\rho(p)^r + O(p^\gamma X^\delta) + O\left(\frac{X}{p}\right) = \frac{|L(X)| n_r}{1+f(p)} + O(p^\gamma X^\delta) + O\left(\frac{X}{p}\right).$$

So if $X > p^{\frac{1+\gamma}{1-\delta}}$,

$$\frac{1}{|L(X)|}\sum_{\rho\in L(X)} a_\rho(p)^r = \frac{n_r}{1+f(p)} + O(p^{-1}).$$

6. Counting S_5 quintic fields with local conditions

Shankar and Tsimerman [16] recently counted S_5 quintic fields with a power saving error terms. For $i=0,1,2$, let $N_5^{(i)}(X)$ be the number of S_5 quintic fields of signature $(5-2i,i)$ with $|d_K|<X$. Then they showed

$$N_5^{(i)}(X)=D_iX+O_\epsilon\left(X^{\frac{399}{400}+\epsilon}\right),$$

where $D_i=d_i\prod_p(1+p^{-2}-p^{-4}-p^{-5})$ and d_0,d_1,d_2 are $\frac{1}{240}$, $\frac{1}{24}$ and $\frac{1}{16}$, respectively.

We can count quintic fields with finitely many local conditions. Let C be a conjugacy class of S_5 and $f(p)=p^{-1}+2p^{-2}+2p^{-3}+p^{-4}$. Let $\mathcal{S}=\{LC_p\}$ be a finite set of local conditions. Define $|\mathcal{S}_{p,C}|=\frac{|C|}{|G|(1+f(p))}$, $|\mathcal{S}_{p,r_i}|=\frac{c_i(p)}{(1+f(p))}$, and $|\mathcal{S}|=\prod_p|LC_p|$, where $c_i(p)$'s are given explicitly at the end of this section.

Theorem 6.1. *Let $N_5^{(i)}(X,\mathcal{S})$ be the number of S_5 quintic fields of signature $(5-2i,i)$ with $|d_K|<X$, and with the local condition $\mathcal{S}$. Then*

$$N_5^{(i)}(X,\mathcal{S})=|\mathcal{S}|D_iX+O_\epsilon\left(\left(\prod_{p\in\mathcal{S}}p\right)^{2-\epsilon}X^{\frac{199}{200}+\epsilon}\right).$$

We follow the notations in [16]. Let $V_{\mathbb{Z}}$ be the space of 4-tuples of 5×5 alternating matrices with integer coefficients. The group $G_{\mathbb{Z}}=GL_4(\mathbb{Z})\times SL_5(\mathbb{Z})$ acts on $V_{\mathbb{Z}}$ via

$$(g_4,g_5)\cdot(A,B,C,D)^t=g_4(g_5Ag_5^t,g_5Bg_5^t,g_5Cg_5^t,g_5Dg_5^t)^t.$$

Here $g_4\cdot(A,B,C,D)^t$ means $(a_1(A,B,C,D)^t,a_2(A,B,C,D)^t,a_3(A,B,C,D)^t,a_4(A,B,C,D)^t)$, where a_i is the ith row of g_4.

There is a canonical bijection between the set of $G_{\mathbb{Z}}$-equivalence classes of elements $(A,B,C,D)\in V_{\mathbb{Z}}$, and the set of isomorphism classes of pairs of (R,R'), where R is a quintic ring and R' is a sextic resolvent ring of R. (See [5].) Let $\mathcal{V}$ be an element of $V_{\mathbb{Z}}$. Over the residue field $\mathbb{F}_p$, the element $\mathcal{V}$ determines a quintic $\mathbb{F}_p$-algebra $R(\mathcal{V})/(p)$. Let us define the splitting symbol $(\mathcal{V},p)$ by

$$(\mathcal{V},p)=(f_1^{e_1}f_2^{e_2}\cdots),$$

whenever $R(\mathcal{V})/(p)\cong\mathbb{F}_{p^{f_1}}[t_1]/(t_1^{e_1})\oplus\mathbb{F}_{p^{f_2}}[t_2]/(t_2^{e_2})\oplus\cdots$. Then there are 17 possible splitting types for $(\mathcal{V},p)$; (11111), (1112), (122), (113), (23), (14), (5), (1^2111), (1^212), (1^23), (1^21^21), (2^21), (1^311), (1^32), (1^31^2), (1^41), and (1^5). Let σ be one of 17 splitting types. Then define $T_p(\sigma)$ to be the set of $\mathcal{V}\in V_{\mathbb{Z}}$ such that $(\mathcal{V},p)=\sigma$ and $U_p(\sigma)$ to be the set of elements in $T_p(\sigma)$ corresponding

to quintic rings that are maximal at p. The set $U_p(\sigma)$ is defined by congruence conditions on coefficients of $\mathcal{V}$ modulo p^2. Let $\mu(U_p(\sigma))$ be the p-adic density of $\mathcal{S}$ in $V_{\mathbb{Z}_p}$. They are computed in Lemma 4 in [5]. Let U_p denote the union of the 17 $U_p(\sigma)$. Then Lemma 20 of [5] implies that

$$\mu(U_p) = (p-1)^8 p^{12}(p+1)^4(p^2+1)^2(p^2+p+1)^2(p^4+p^3+p^2+p+1)(p^4+p^3+2p^2+2p+1)/p^{40}.$$

Note that

$$d_i\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)\prod_p \mu(U_p) = d_i\prod_p(1+p^{-2}-p^{-4}-p^{-5}),$$

which is the coefficient of the main term in counting quintic fields. Here we need to interpret $\mu(U_p)$ in the following way: U_p can be considered as a subset of $(\mathbb{Z}/q^2\mathbb{Z})^{40}$, or the union of k translates of $p^2V_{\mathbb{Z}}$, where k is the size of the set. Here k is $\mu(U_p)q^{80}$. Let W_p be the complement of U_p in $V_{\mathbb{Z}}$, then $\mu(W_p) = 1-\mu(U_p)$. Then W_p is the union of $\mu(W_p)q^{80}$ translates of $q^2V_{\mathbb{Z}}$.

For q square-free, let $W_q \subset V_{\mathbb{Z}}$ be the set of elements corresponding to quintic rings that are not maximal at each prime dividing q. Then W_q is the union of $\prod_{p|q}\mu(W_p)\cdot q^{80}$ translates of $q^2V_{\mathbb{Z}}$ by the Chinese Remainder Theorem.

Let $V_{\mathbb{Z}}^{ndeg}$ denote the set of elements in $V_{\mathbb{Z}}$ that correspond to orders in S_5-fields, and let $V_{\mathbb{Z}}^{deg}$ be the complement of $V_{\mathbb{Z}}^{ndeg}$. A point in $V_{\mathbb{Z}}$ corresponds to a maximal order in an S_5 quintic fields precisely if it is in $\cap_p U_p \cap V_{\mathbb{Z}}^{ndeg}$. For a $G_{\mathbb{Z}}$-invariant subset S of $V_{\mathbb{Z}}$, let $N(S,X)$ denote the number of irreducible $G_{\mathbb{Z}}$-orbits in $S^{ndeg} := S\cap V_{\mathbb{Z}}^{ndeg}$ having discriminant bounded by X. For a set S which is not $G_{\mathbb{Z}}$-invariant, we can define $N^*(S,X)$ which also counts the orbits of degenerate points in S.

Now we choose a finite set of primes $\{p_1,p_2,\cdots,p_n\}$. And choose a splitting type σ_{p_k} for each p_k, $k=1,2,\cdots,n$. Define U_p' to be U_p if $p\neq p_k$, $k=1,2,\cdots,n$. If $p=p_k$ for some k, then $U_p' = U_p(\sigma_{p_k})$. Let W_p' be the complement of U_p'. Then W_q' is the union of $\prod_{p|q}\mu(W_p')\cdot q^{80}$ translates of $q^2V_{\mathbb{Z}}$.

Let $N_5^{(i)}(X,\{\sigma_{p_k}\}_{k=1,2,\cdots,n})$ be the number of S_5 quintic fields of signature $(5-2i,i)$ with $|d_K|<X$ and the splitting types of p_k's are σ_{p_k} for $k=1,2,\cdots,n$. Then by inclusion-exclusion method,

$$N_5^{(i)}(X,\{\sigma_{p_k}\}_{k=1,2,\cdots,n}) = \sum_q \mu(q)N(W_q'\cap V_{\mathbb{Z}}^{(i)},X).$$

We use two estimates of $N(W_q'\cap V_{\mathbb{Z}}^{(i)},X)$ for small q's and large q's separately. Lemma 3 in [16] says that $N(W_q,X) = O_\epsilon(X/q^{2-\epsilon})$ and it implies that $N(W_q',X) \ll_\epsilon (p_1p_2\cdots p_n)^{2-\epsilon}\frac{X}{q^{2-\epsilon}}$ for

q square-free. If L is a translate of $mV_{\mathbb{Z}}$, then we have

$$N^*(\{x \in L \cap V_{\mathbb{Z}}^{(i)} : a_{12} \neq 0\}, X) = c_i \frac{X}{m^{40}} + O\left(m^{-39} X^{\frac{39}{40}}\right), \tag{6.1}$$

where $c_i = d_i \zeta(2)^2 \zeta(3)^2 \zeta(4)^2 \zeta(5)$. (See Equation 28 in [6].) Since W_q' is an union of $\prod_{p|q} \mu(W_p') \cdot q^{80}$ translates of $q^2 V_{\mathbb{Z}}$,

$$N^*(\{x \in W_q' \cap V_{\mathbb{Z}}^{(i)} : a_{12} \neq 0\}, X) = \prod_{p|q} \mu(W_p') \cdot c_i X + O\left(q^2 X^{\frac{39}{40}}\right) = \mu(W_q') \cdot c_i X + O\left(q^2 X^{\frac{39}{40}}\right).$$

Then

$$\begin{aligned}
&N_5^{(i)}(X, \{\sigma_{p_i}\}_{i=1,2,\cdots,n}) = \sum_q \mu(q) N(W_q' \cap V_{\mathbb{Z}}^{(i)}, X) \\
&= \sum_{q \leq Q} \left(\mu(W_q') \cdot c_i X + O\left(q^2 X^{\frac{39}{40}}\right) - \mu(q) N_{12}^*(W_q \cap V_{\mathbb{Z}}^{deg,(i)}, X)\right) + \sum_{q > Q} O_\epsilon\left((p_1 p_2 \cdots p_n)^{2-\epsilon} \frac{X}{q^{2-\epsilon}}\right) \\
&= \sum_{q \leq Q} \left(\mu(W_q') \cdot c_i X - \mu(q) N_{12}^*(W_q \cap V_{\mathbb{Z}}^{deg,(i)}, X)\right) + O_\epsilon\left((p_1 p_2 \cdots p_n)^{2-\epsilon} X / Q^{1-\epsilon} + X^{\frac{39}{40}} Q^{3+\epsilon}\right) \\
&= \sum_q c_i \mu(W_q') X + (p_1 p_2 \cdots p_n)^{2-\epsilon} O_\epsilon\left(X/Q^{1-\epsilon} + X^{\frac{39}{40}} Q^{3+\epsilon} + X^{\frac{199}{200}} Q^{1+\epsilon}\right) \\
&= c_i \prod_p (1 - \mu(W_q')) X + (p_1 p_2 \cdots p_n)^{2-\epsilon} O_\epsilon\left(X/Q^{1-\epsilon} + X^{\frac{39}{40}} Q^{3+\epsilon} + X^{\frac{199}{200}} Q^{1+\epsilon}\right) \\
&= c_i \prod_p (\mu(U_q')) X + (p_1 p_2 \cdots p_n)^{2-\epsilon} O_\epsilon\left(X/Q^{1-\epsilon} + X^{\frac{39}{40}} Q^{3+\epsilon} + X^{\frac{199}{200}} Q^{1+\epsilon}\right).
\end{aligned}$$

Putting $Q = X^{\frac{1}{400}}$, we have

$$(p_1 p_2 \cdots p_n)^{2-\epsilon} O_\epsilon\left(X/Q^{1-\epsilon} + X^{\frac{39}{40}} Q^{3+\epsilon} + X^{\frac{199}{200}} Q^{1+\epsilon}\right) \ll_\epsilon (p_1 p_2 \cdots p_n)^{2-\epsilon} X^{\frac{399}{400}+\epsilon}.$$

Note that

$$c_i \prod_p (\mu(U_q')) \quad = \quad \prod_{k=1}^n \frac{\mu(U_p(\sigma_{p_k}))}{\mu(U_{p_k})} c_i \prod_p \mu(U_p) = \prod_{k=1}^n \frac{\mu(U_p(\sigma_{p_k}))}{\mu(U_{p_k})} d_i \prod_p \left(1 + p^{-2} - p^{-4} - p^{-5}\right).$$

From Lemma 20 in [5], we can see that, for $f(p) = p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4}$,

$$\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/120}{1+f(p)}, \frac{1/12}{1+f(p)}, \frac{1/8}{1+f(p)}, \frac{1/6}{1+f(p)}, \frac{1/6}{1+f(p)}, \frac{1/4}{1+f(p)}, \text{ and } \frac{1/5}{1+f(p)}$$

for $\sigma = (11111), (1112), (122), (113), (23), (14), (5)$, respectively, and

$$\begin{aligned}\frac{\mu(U_p(\sigma))}{\mu(U_p)} \quad &= \quad \frac{1/6\cdot 1/p}{1+f(p)}, \frac{1/2\cdot 1/p}{1+f(p)}, \frac{1/3\cdot 1/p}{1+f(p)}, \frac{1/2\cdot 1/p^2}{1+f(p)}, \frac{1/2\cdot 1/p^2}{1+f(p)}, \frac{1/2\cdot 1/p^2}{1+f(p)}, \frac{1/2\cdot 1/p^2}{1+f(p)}, \\ &\quad\ \frac{1/p^3}{1+f(p)}, \frac{1/p^3}{1+f(p)}, \ \text{and } \frac{1/p^4}{1+f(p)}\end{aligned}$$

for $\sigma = (1^2111), (1^212), (1^23), (1^21^21), (2^21), (1^311), (1^32), (1^31^2), (1^41), (1^5)$, respectively. Hence we have proved the theorem.

7. Counting S_4 quartic fields with local conditions

In [3], Belabas, Bhargava and Pomerance obtained a power saving error term for counting S_4 quartic fields. For $i = 0, 1$, let $N_4^{(i)}(X)$ be the number of isomorphism classes of S_4-quartic fields of signature $(4-2i, i)$ with $|d_K| < X$. Then

$$N_4^{(i)}(X) = D_iX + O(X^{23/24+\epsilon}),$$

where $D_i = d_i \prod_p (1 + p^{-2} - p^{-3} - p^{-4})$, and $d_0 = \frac{1}{48}, d_1 = \frac{1}{8}$, and $d_0 = \frac{1}{16}$.

Let $V_{\mathbb{Z}}$ be the space of pairs of (A, B) of integral ternary quadratic forms. The group $G_{\mathbb{Z}} = GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ acts on $V_{\mathbb{Z}}$ in the following way. For $g_2 \in GL_2(\mathbb{Z})$ and $g_3 \in SL_3(\mathbb{Z})$,

$$(g_2, g_3) \cdot (A, B)^t = g_2(g_3Ag_3^t, g_3Bg_3^t)^t.$$

Here $g_2 \cdot (A, B)$ means that $(a_1(A, B)^t, a_2(A, B)^t)$, where a_i is the ith row of g_2.

There is a canonical bijection between the set of $G_{\mathbb{Z}}$-equivalence classes of elements $(A, B) \in V_{\mathbb{Z}}$, and the set of isomorphism classes of pairs of (Q, R), where Q is a quartic ring and R' is a cubic resolvent ring of Q. (See [4].)

A prime p has 11 possible splitting type in an S_4 quartic field K. They are (1111), (112), (13), (4), (22), (1^211), (1^21^2), (1^22), (2^2), (1^31), and (1^4). As we did in the previous section, we can define $U_p(\sigma)$ and U_p, resp. and $\mu(U_p(\sigma))$'s are computed in Lemma 23, [4].

Using their result, Yang [18] was able to count S_4 quartic fields with one local condition with a power saving error term. He showed that

$$N_4^{(i)}(X, \sigma_p) = \frac{\mu(U_p(\sigma_p))}{\mu(U_p)} D_iX + O\left(p^2X^{\frac{143}{144}+\epsilon}\right).$$

By the same argument in the previous section, we can extend Yang's result to the case of finitely many local conditions:

$$N_4^{(i)}(X, \{\sigma_{p_k}\}_{k=1,2,\cdots,n}) = \left(\prod_{k=1} \frac{\mu(U_p(\sigma_{p_k}))}{\mu(U_{p_k})}\right) D_i X + O\left((p_1 \cdots p_k)^2 X^{\frac{143}{144}+\epsilon}\right).$$

From Lemma 23 in [4], we can see that, for $f(p) = p^{-1} + 2p^{-2} + p^{-3}$,

$$\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/24}{1+f(p)}, \ \frac{1/4}{1+f(p)}, \ \frac{1/3}{1+f(p)}, \ \frac{1/8}{1+f(p)}, \text{ and } \frac{1/4}{1+f(p)}$$

for $\sigma = (1111), (112), (13), (22), (4),$ respectively, and

$$\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/2 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/p^2}{1+f(p)}, \text{ and } \frac{1/p^3}{1+f(p)}$$

for $\sigma = (1^2 11), (1^2 2), (1^2 1^2), (2^2), (1^3 1), (1^4)$, respectively. Hence we have proved

Theorem 7.1. *Let $N_4^{(i)}(X, \mathcal{S})$ be the number of S_4 quartic fields of signature $(4-2i, i)$ with $|d_K| < X$, and with the local condition $\mathcal{S}$. Then*

$$N_4^{(i)}(X, \mathcal{S}) = |\mathcal{S}| D_i X + O_\epsilon\left(\left(\prod_{p \in \mathcal{S}} p\right)^2 X^{\frac{143}{144}+\epsilon}\right).$$

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Department of Mathematics, State University of New York at Buffalo, NY 14260, USA

E-mail address: `jcho23@buffalo.edu`

Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, CANADA, and Korea Institute for Advanced Study, Seoul, Korea

E-mail address: `henrykim@math.toronto.edu`